4101. (a) Find the first and second derivatives. Show that the second derivative is zero and changes sign at the roots of $\sin x$. Show that half of these (every other point) has gradient 0.

(b) The graph is a superposition of $y = x$ and $y = \sin x$, so it is a sinusoidal oscillation about the line $y = x$. The results of part (a) tell you exactly the extent of the oscillation.

4102. Find the angles of inclination of the planes, and hence the angles of inclination of the reaction forces acting on the cylinder. Draw a triangle of forces, and express R_1 and R_2 in terms of W.

4103. Integrate by parts. Let $u = x$ and $v' = \text{f}_2(x)$.

4104. Differentiate twice, and take out a common factor of x from the second derivative. Show, therefore, that the second derivative is zero and changes sign at $x = 0$.

4105. Consider the fact that, in order for a triangle to exist, each side must be shorter than the sum of the other two.

4106. All of the points of the given graph have $y \geq 0$. So, they all feature in the new graph. And replacing y with $|y|$ also allows negative values of y.

4107. Use the substitution $u = e^x + 3$.

4108. (a) Set up a generic tangent through $\big(p, e^p + e^{-p}\big)$. Substitute $(0, 0)$ into this and rearrange.

(b) The equation in (a) isn't analytically solvable. Use N-R (or fixed-point iteration).

4109. Use a double-angle formula to express $\sin^2 x$ in terms of $\cos 2x$.

4110. This can be done by symmetry, without calculus.

4111. Use the symmetry of the problem to establish that one of the roots must be at $x = 0$. This will give you the value of a. Set up an equation to solve for intersections, and require that it has precisely three roots.

4112. Since the GP is increasing and the common ratio $r > 1$, all terms must be positive. Express them as a, ar, ar^2, ar^3, ar^4. Then consider

① $b + d - 2c$,

② $a + e - (b + d)$.

Factorise to show that each is +ve.

4113. Find the intersection of the curves in terms of m. Then set up a definite integral in terms of m, and equate it to 36. Solve for m.

4114. Find the probability that the first two sampled are above the 90th percentile and the other two are not. Then multiply this by the number of orders of AABB.

4115. Two of these are linear transformations (consisting of stretches and translations). Hence, their images do each contain a point of inflection. The image of the nonlinear transformation doesn't.

4116. Set up a unit circle, without loss of generality, with one of the vertices of the rectangle at $(\cos\theta, \sin\theta)$. Express the area of the circle in terms of θ, and simplify with an identity. Show that the area is maximised at $\theta = 45°$. Alternatively, a (carefully phrased) symmetry argument will work.

4117. Integrate by parts, with $u = \ln x$.

4118. Show that the locus contains the unit circle in the positive quadrant, but that it doesn't contain the unit circle in the negative quadrant.

4119. Notice that both sides have a factor of $x^{\frac{1}{2}} + 3$.

— Alternative Method —

Substitute $u = \sqrt{x}$ to produce a cubic in u.

4120. (a) Differentiate by the quotient rule.

(b) Divide top and bottom by x^2 before taking the limit $x \to \pm\infty$.

(c) Use part (b) to draw the horizontal asymptote. Establish that there is no vertical asymptote. Find the x intercepts.

4121. (a) The integrand, as a function, is

$$z \longmapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

Plug z values $\{0, 0.25, 0.5, 0.75, 1\}$ into this. Then use the trapezium rule formula.

(b) The standard normal distribution has points of inflection at $z = \pm 1$. It is concave on $(-1, 1)$ and convex on $(-\infty, -1) \cup (1, \infty)$.

4122. (a) Establish that the sloped distance from the ground to the top of the wall is $l \operatorname{cosec}\theta$. Use this to take moments, and cancel a factor of $l\cos\theta$. Then multiply both sides by $\sin\theta$.

(b) Take moments around the base of the ladder. Write R_2 in terms of W, and use a double-angle formula to simplify. Then quote the range of $\sin 2\theta$.

4123. Trying to do this algebraically, i.e. by solving for intersections, gets very tough quickly. Instead, since you don't have to find the re-intersection, merely show it exists, use a graphical approach. Sketch $y = xe^x$ in detail, and both parts become relatively obvious.

4124. Equate the expressions for area. Then double and square both sides. Rearrange this into the required form and use the standard Pythagorean theorem.

— Alternative Method —

Find expressions for $|AD|$ and $|BD|$. Then write $|AD| + |BD| = c$. Square both sides and simplify to the required form.

4125. Write the infinite bound as a limit:

$$\int_1^\infty \frac{\ln x + 1}{x^2}\,dx = \lim_{k\to\infty}\int_1^k \frac{\ln x + 1}{x^2}\,dx.$$

Then integrate by parts, using $u = \ln x + 1$.

4126. Solve the DE by separation of variables. Use the origin to produce the particular solution. Show that $(\pm\sqrt{3}, 0)$ are points on it.

4127. Make a table of the probability distribution of $H \sim B(4, 1/2)$, and add probabilities.

4128. (a) Consider line L through (x_0, y_0) and (x_1, y_1), with reference to similar triangles. Show that the other centres must lie on it.
(b) Find the angle between line L and the x axis.

4129. Compare values in the following table:

f(0)	f′(0)	f′′(0),
g(0)	g′(0)	g′′(0).

4130. (a) Sketch the boundary cases, in which edges of the two squares coincide. Then use symmetry.
(b) Consider the midpoint of the interval found in part (a).

4131. Let $u = \ln|\sin x|$ and $\frac{dv}{dx} = \sec^2 x$. The derivative of the former and the integral of the latter are both (fairly) standard results. The derivative is better known as an integral result, and the integral is better known as a derivative result.

4132. (a) Substitute in a and b to the LHS and expand by the cosine compound-angle formulae. Then solve simultaneously to find a and b in terms of x and y. Substitute these results back in at the end.
(b) Consider the sum $\cos pt + \cos qt$. Rewrite this using the identity from (a). Then consider the values of $\frac{p+q}{2}$ and $\frac{p-q}{2}$, as compared to the original p and q.

4133. Determine the total shaded area by evaluating

$$\lim_{k\to\infty}\int_2^k (x-1)^{-2}\,dx.$$

Find the gradient of the tangent, and therefore the area of the triangle below it.

4134. Solve for intersections of $y = U_n$ and $y = V_n$: take them as continuous graphs. Sketch them, noting that the leading coefficient of U_n is larger than that of V_n.

4135. The even and odd cases are different.

4136. Substitute the latter into the former, and use log rules to reach a cubic in x. Find a root, and use it to factorise. Show that the remaining quadratic has no real roots.

4137. Find the first and second derivative, factorising the first derivative before differentiating it again.
(a) Show that the second derivative is positive.
(b) Show that the second derivative is zero and changes sign.

4138. Solve to find the x coordinates x_1 and x_2 of P and Q, in terms of k. Hence, find an expression for the difference $x_2 - x_1$. Multiply this by $\sqrt{2}$ to give $|PQ|$, and equate this to the given value.

4139. Either use the conditional probability formula, or count outcomes in the restricted possibility space.

4140. Put the fractions over a common denominator. Then consider the behaviour as $x \to 1^{\pm}$. Also, consider the behaviour as $x \to \pm\infty$. Then use the quotient rule to find stationary points.

4141. (a) Set the first derivative to zero.
(b) The right-hand stationary point, if it exists, must be above the x axis. Use this to show that $-2k^{\frac{3}{2}} + 2 > 0$. Solve.

4142. Consider the multiplicity of the roots of f.

4143. Convexity implies that each interior angle must satisfy $\theta \in (0, \pi)$. Find the mean of the AP. Then consider the fact that this mean is closer to π than to 0. The common difference is maximised when the largest angle is at (the unattainable bound) π.

4144. (a) Set the denominator to zero. The equation isn't analytically solvable, so use a numerical method. N-R tends to be most efficient.

(b) Differentiate by the quotient rule, and set the numerator to zero for SPs.

4145. You don't need to use the trapezium rule at all here. Just show that the function $x \mapsto \tan(x^2)$ is convex on the domain $(0, 1)$. For the second derivative, use the result $\frac{d}{dx}\sec^2 x = 2\sec^2 x \tan x$.

4146. Find the intersections of the curves. Test these points in the LHS of the equation of the ellipse.

4147. (a) i. Find the tension in the cable. Then draw a force diagram for one of the pulleys. Two tensions should act. Add by Pythagoras, or give the force as a vector.

ii. Use NII on the load to find the new tension, and then repeat the above. Note that the combined horizontal forces cancel. Also, since the pulleys are light, the combined downwards force on the pulleys is the same as a combined downwards force on the arm.

(b) Use the method of (a) ii. to find the maximum safe tension in the cable. Then use NII for the load to find the acceleration.

4148. Rotation by 180° about the origin is the same as reflection in both x and y axes. Consider these individual transformations as input replacements.

4149. The inequality describes the interior of an ellipse. Show that boundary ellipse and parabola have no intersections. This implies that the parabola lies outside the ellipse.

4150. At a point of inflection, the curvature (second derivative) changes continuously from +ve to −ve, or vice versa. For this, it is necessary, but not sufficient, for the second derivative to be zero.

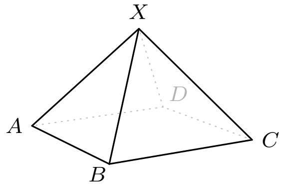
4151. (a) Substitute expressions for x and y into the LHS, and simplify using trig identities.

(b) If the distance from the curve to the y axis is stationary with respect to t, then $\frac{dx}{dt} = 0$.

4152. Two separate binomial expansions would be hard work. Instead, take out a factor of $^1/_{x^4}$ from the right-hand bracket, and combine the factors as a difference of two squares. Then expand binomially.

4153. Establish the equations of the relevant tangents are $y = 2ax - a^2$ and $y = 2(a-3)x - (a-3)^2$. Substitute the point $(b, -2)$ into each to get a pair of simultaneous equations in a and b.

4154. The centres of the four spheres A, B, C, D and the point from which the strings are suspended X form a square based pyramid, in which all edges lengths are $2r$.



Resolve vertically in this picture.

4155. Take this bit by bit. Firstly, simplify $(x^2 + y^2)$ on its own. Then substitute this and the formulae for x and y into the LHS of the Cartesian equation. Take a common factor of $4(1 - \cos t)^2$ out, then use the first Pythagorean trig identity.

4156. Both the LHS and the RHS have even symmetry. So, we need only show the result for non-negative x. Since $(1+x^2)$ is positive, you can multiply by it, while maintaining the direction of the inequality.

4157. The y axis is the line of symmetry of the $y = x^2$. Find the angle between the y axis and $y = \sqrt{3}x$. Use this to find the new angle of inclination θ, above the x axis, of the line of symmetry. The gradient is then given by $m = \tan\theta$.

4158. Write $v = x^2$ as $\frac{dx}{dt} = x^2$. Solve this DE in x and t by separation of variables. Then consider the t value at which the position grows without bound.

4159. The relevant fact is the parity of the degree of f.

4160. Use the first Pythagorean identity to form a cubic in $\sin x$. Substitute $z = \sin x$. Use a polynomial solver to find the roots of the resulting cubic, and so factorise it. Show that no x values satisfy the resulting equations.

4161. Differentiate implicitly to find the first derivative as a fraction in terms of x, y. You can differentiate again, but it's a bit fiddly. It's easier to show that there is no change of sign in the gradient.

4162. The first three are standard results. For the fourth, write down the distribution of $aX_i + b$ first, before using the result of part (b).

4163. Find, in terms of θ, the gradient of the radius, then the gradient of the tangent, then the equation of the tangent. Use this to find the axis intercepts A and B. Set up an expression for the area of OAB and simplify it with a double-angle formula.

——— ALTERNATIVE METHOD ———

Drop a perpendicular from P to OA. Call that point X. Then use the fact that triangles OAP, OBP and OXP are all similar to find $|OA|$. $|OB|$ follows by symmetry.

4164. Divide top and bottom by the common factor x. Then write the integrand as a proper algebraic fraction (degree of numerator lower than degree of denominator). Finally, put it in partial fractions.

4165. Set up the equation algebraically. Then rearrange to the form $\text{fg}(x) = \text{gf}(x)$ and square both sides.

4166. (a) Rearrange to the form $v = A + (...)e^{-kt}$.
(b) A and B are velocities, k is a rate.
(c) Consider the fact that $A < B$.
(d) B has the same units as A. k has units so as to ensure that the inputs into the exponential function are unitless.
(e) Integrate the rearranged form of the velocity, including the $+c$. Set $s = 0$ at $t = 0$. As often happens with exponentials, this gives a non-zero constant of integration.

4167. Use the change of base formula

$$\log_a b = \frac{\log_c b}{\log_c a}$$

to write the logarithms as natural logarithms. Hence, write $\text{f}(x) = k \ln x$, giving k in terms of a and b. Use the fact $1 < a < b$ and the properties of the ln function to show that k must be positive.

4168. (a) Find the gradient at the origin.
(b) You need to show that the line $y = -x$ is not normal to $y = \tan x$ at the first positive (or negative) point of intersection. Solve for such an intersection using a numerical method, and then test the gradient of $y = \tan x$.

4169. The question here is the direction from which one approaches 1. Consider the cases $x \to 1^+$ and $x \to 1^-$ separately. If they give the same result, then the limit is well defined as written.

4170. Call the cubics $\text{f}(x)$ and $\text{g}(x)$, and consider the equation $\text{f}(x) - \text{g}(x) = 0$.

4171. (a) Find the axis intercepts of each curve.
(b) Find the cross-sectional area of the water at A, by definite integration. You'll need the area between a line (the surface of the water) and the curve.
Then set up the same procedure at B, this time with a known area and an unknown line. Use $\pm k$ as the x coordinates of the edges of the water.

4172. Write $\text{f}(x)$ in harmonic form. Sketch a transformed sine wave, locating the largest interval containing zero over which the curve is one-to-one. You don't need calculus to find the SPs.

4173. This is a cubic in e^x. Take out a factor of e^x, which is always positive, and show that the remaining quadratic in e^x is also always positive.

4174. By Pythagoras, $x^2 + y^2 = r^2$. Also (noting reversal of sin and cos due to an angle with the y axis),

$$x = r \sin\theta,$$
$$y = r \cos\theta.$$

Substitute these in and simplify.

4175. (a) Use your calculator.
(b) Show that the z value is 1.645. Then use

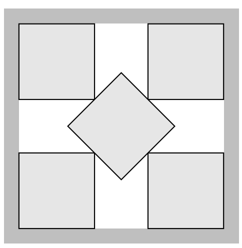
$$z = \frac{x - \mu}{\sigma}.$$

4176. There are two things to prove here. Firstly, that for $x > 0$, the integral of $\frac{1}{x}$ is $\ln x + c$. To do this, write $x \equiv e^{\ln x}$ and differentiate both sides with respect to x. Secondly, that for $x < 0$, the integral of $\frac{1}{x}$ is $-\ln x + c$. To do this, write $x \equiv -e^{\ln(-x)}$ and differentiate both sides with respect to x.

4177. (a) This is a positive sextic with a single root at $x = a$, a double root at $x = b$ and a triple root at $x = c$.
(b) The graph has double roots at $x = a$ and $x = c$, and a double asymptote at $x = b$. In its behaviour for large x, it is close to parabolic.

4178. (a) Consider that you are only given information about the second derivative.
(b) Write down the set of x values for which the second derivative, i.e. the y value on the graph $y = \text{g}''(x)$, is positive.
(c) At a point of inflection, the second derivative is zero and changes sign.

4179. Show that the squares fit exactly in the following configuration:



4180. Differentiate implicitly and set $\frac{dy}{dx} = 0$. Simplify and substitute back into the given equation.

4181. (a) Set up the equality (boundary equation).
  (b) Follow the same algebraic path as in (a), but in reverse. Start with $(a-b)^2 \geq 0$.

4182. (a) Show that the gradient is non-negative over the domain, and that the function is therefore one-to-one.
  (b) Show that there is a turning point (minimum) in the domain $[0, \pi/2)$, and that the function is therefore not one-to-one.

4183. Set $\dot{x} = 0$ for tangents parallel to y, of the form $x = k$. Solve these simultaneously with the curve to find the coordinates of P and Q.

4184. (a) Draw a force diagram, with tensions T_1 and T_2. Resolve horizontally to show that $T_1 \neq T_2$.
  (b) Horizontally, the sections of wire are
    i. asymmetrical, but
    ii. very nearly symmetrical.
  (c) No calculation is needed here.

4185. Take out a common factor of $(x+2)^3$. Using the fact that a fourth power must be non-negative, show that the remaining quartic equation has no real roots.

4186. Integrate with respect to x. To do this, use the substitution $u = e^x + 2$, then write the integrand in partial fractions.

4187. Show that $x^2y + xy^2 = 1$ cannot contain any points in the third quadrant, which the graph does.

4188. (a) Find the second derivative and substitute into the DE.
  (b) Find the second derivative, using the product rule, and substitute into the DE. You should find that two of the terms cancel. Divide through by the non-zero exponential, and then integrate your equation with respect to x.
  (c) Let $z = \mathrm{f}(x)$. Then write $\frac{dz}{dx} = k - 4z$ and solve by separation of variables. Rename the constants that arise, possibly more than once.
  (d) Substitute the result from (c) back into the original solution $y = \mathrm{f}(x)e^{2x}$, and simplify.

4189. The implication goes forwards. Prove this, then consider $\mathrm{f}'(x) = (x-1)^2 + 1$ as a counterexample to the backwards implication.

4190. Prove that triangles AXD and BXC are similar.

4191. This is a separable DE. Separate the variables and integrate. Before rearranging, sub in the initial conditions to find the constant of integration.

4192. (a) Note that the inverse of arcsin is not the full sine function, but rather the restricted version used to generate arcsin in the first place.
  (b) Consider point $(\sin x, x)$ on $y = \mathrm{f}(x)$. Reflect this in $y = x$, then find the gradient. Reflect this back to find the original gradient.
  (c) Use the first Pythagorean trig identity, being careful to justify the positive square root.

4193. Represent the restricted possibility space by filling in the gaps in the following table.

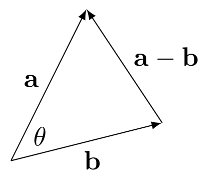
Values	Total	Successful
$(1,5,6)$	$3!$	0
$(2,4,6)$	$3!$	*
$(2,5,5)$	*	*
*	*	*
*	*	*
*	*	*

The total in the second column is the number of orders of the values in the first column.

4194. Find the x intercepts. Then set up a sequence of definite integrals. Integrate by parts.

4195. Let the equation of the cubic be $y = \mathrm{f}(x)$ and the equation of the line be $y = \mathrm{g}(x)$. Call the point of inflection $x = \alpha$. Consider the multiplicity of the root $x = \alpha$ of the equation $\mathrm{f}(x) - \mathrm{g}(x) = 0$.

4196. (a) Use Pythagoras, expressing both $\sec\phi$ and $\tan\phi$ in terms of $\sin\phi$ and $\cos\phi$.
  (b) Use the results of part (a) in the cosine rule on the following vector triangle:



4197. Consider writing $\mathrm{f}(\theta)$ in harmonic form. You don't need to find α, only to consider the amplitude R. Draw a graph of $y = R\sin\theta$,

4198. Call the roots x_1, x_2 for the first equation and x_3, x_4 for the second equation. Find expressions for $x_1 + x + 2$ and $x_3 + x_4$ using the quadratic formula or otherwise. Then add your expressions.

4199. (a) Setting up natural axes, find the coordinates of the midpoint of the trapdoor, when at angle of inclination θ. Then use Pythagoras.

(b) Differentiate the equation from (a) implicitly, with respect to time. The constant speed u of retraction allows you to substitute in for $\frac{dl}{dt}$. Choose the sign carefully. Simplify with

$$\cos\theta = \sqrt{1 - \sin^2\theta}.$$

(c) The speed of opening is maximised when $1 - l^2$ is minimised.

4200. The constant of integration has appeared too late.

——— End of 42nd Hundred ———